CALCULUS 1E ASSIGNMENT 1

PART 1

a) Prove using mathematical induction that:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

For n = 1

$1^2 = \frac{1}{6}(1+1)(2+1)$

$1 = \frac{1}{6}\times 2\times 3$

1 = 1

For n = n + 1

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + ... + n^2 + (n+1)^2 = \frac{1}{6}n(n+1)(n+2)(2n+3)$$

$LHS = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$

$\frac{1}{6}(n+1)[n(2n+1) + 6(n+1)]$

$\frac{1}{6}(n+1)(2n^2 + n + 6n + 6)$

$\frac{1}{6}(n+1)(2n^2 + 7n + 6)$

$\frac{1}{6}(n+1)(n+2)(2n+3) = RHS$ QED

Use the formula above to:

(I) Find the value of $1^2 + 2^2 + 3^2 + \ldots + 10^2$

$$\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + ... + 10^2 = \frac{1}{6}n(n+1)(2n+1)$$

$\frac{1}{6}\times 10(10+1)(20+1)$

$\frac{1}{6}\times 10\times 11\times 21$

= 385

(ii) Find an expression for $\left(\frac{a}{4}\right)^2+\left(\frac{2a}{4}\right)^2+\left(\frac{3a}{4}\right)^2+....+\left(\frac{ra}{4}\right)^2$

$$\sum_{i=1}^{r}\left[i\left(\frac{a}{4}\right)\right]^2=\left(\frac{a}{4}\right)^2+\left(\frac{2a}{4}\right)^2+\left(\frac{3a}{4}\right)^2+....+\left(\frac{ra}{4}\right)^2=\left(\frac{a}{4}\right)^2\left[\frac{1}{6}r(r+1)(2r+1)\right]$$

b) Develop a formula for the following

$$\sum_{i=1}^{n}(2i-1)^3=1^3+3^3+5^3+....+(2n-1)^3$$

n	1	2	3	4	5	6	7	8	9	10
	1^3	3^3	5^3	7^3	9^3	11^3	13^3	15^3	17^3	19^3
	1	28	153	496	1225	2556	4753	8128	13041	19900
	$1^2\times 1$	$2^2\times 7$	$3^2\times 17$	$4^2\times 31$	$5^2\times 49$	$6^2\times 71$				

$=n^2(2n^2-1)$

Proof by mathematical induction

$$\sum_{i=1}^{n}(2i-1)^3=1^3+3^3+5^3+....+(2n-1)^3=n^2(2n^2-1)$$

For n = 1

$1^3=1^2(2\times 1^2-1)$
$1=2-1$
$1 = 1$

For n = n + 1

$$\sum_{i=1}^{n}(2i-1)^3=1^3+3^3+5^3+....+[2(n+1)-1]^3=n^2(2n^2-1)$$

$(2n+2-1)^3$
$(2n+1)^3$

$$\sum_{i=1}^{n}(2i-1)^3=1^3+3^3+5^3+....+(2n+1)^3=(n+1)^2\left[2(n+1)^2-1\right]$$

$(n^2+2n+1)(2n^2+4n+2-1)$

$(n^2+2n+1)(2n^2+4n+1)$

$2n^4+4n^3+n^2+4n^3+8n^2+2n+2n^2+4n+1$

$RHS = 2n^4+8n^3+11n^2+6n+1$

$\sum_{i=1}^{n}(2i-1)^3 = 1^3+3^3+5^3+....+(2n+1)^3 = 2n^4+8n^3+11n^2+6n+1$

$LHS = n^2(2n^2-1)+(2n+1)^3$

$2n^4-n^2+(2n+1)(2n+1)(2n+1)$

$2n^4-n^2+(4n^2+2n+2n+1)(2n+1)$

$2n^4-n^2+(4n^2+4n+1)(2n+1)$

$2n^4-n^2+8n^3+4n^2+8n^2+4n+1+2n$

$2n^4-n^2+8n^3+12n^2+6n+1$

$2n^4+8n^3+11n^2+6n+1 = RHS$ QED

PART 2

Cylinders inside cone

$V_{3\text{ cylinders}} = \pi\left(\frac{h}{4}\right)\left(\frac{r}{4}\right)^2+\pi\left(\frac{h}{4}\right)\left(\frac{r}{2}\right)^2+\pi\left(\frac{h}{4}\right)\left(\frac{3r}{4}\right)^2$

$\pi\frac{h}{4}\left[\left(\frac{r}{4}\right)^2+\left(\frac{r}{2}\right)^2+\left(\frac{3r}{4}\right)^2\right]$

$\pi\frac{h}{4}\left(\frac{r^2}{16}+\frac{r^2}{4}+\frac{9r^2}{16}\right)$

$\pi\frac{h}{4}r^2\left(\frac{1}{16}+\frac{1}{4}+\frac{9}{16}\right)$

$\pi\left(\frac{h}{4}\right)r^2\times\frac{7}{8}$

$\pi\left(\frac{7h}{32}\right)r^2$

$\pi\frac{7}{32}hr^2 = \frac{7}{32}\pi hr^2$

The volume of a cone is $\frac{1}{3}\pi r^2h$

$\frac{1}{3}\pi r^2h > \frac{7}{32}\pi r^2h$

Cylinders outside cone

$V_{4\text{ cylinders}} = \pi\left(\frac{h}{4}\right)\left(\frac{r}{4}\right)^2 + \pi\left(\frac{h}{4}\right)\left(\frac{r}{2}\right)^2 + \pi\left(\frac{h}{4}\right)\left(\frac{3r}{4}\right)^2 + \pi\left(\frac{h}{4}\right)r^2$

$\pi\frac{h}{4}\left(\frac{r^2}{16} + \frac{r^2}{4} + \frac{9r^2}{16} + r^2\right)$

$\pi\left(\frac{h}{4}\right)r^2\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right)$

$\pi\left(\frac{h}{4}\right)r^2 \times 1\frac{7}{8}$

$\pi\left(\frac{h}{4}\right)r^2 \times \frac{15}{8}$

$\pi\left(\frac{15h}{32}\right)r^2$

$\frac{15}{32}\pi h r^2$

The volume of a cone is $\frac{1}{3}\pi r^2 h$

$\frac{1}{3}\pi r^2 h < \frac{15}{32}\pi h r^2$

Investigate how accurate these rules are for cylinders of particular sizes

Cylinders inside cones

Cone of height 1 and radius 2

$V_{approx} = \frac{7}{32}\pi \times 1 \times 2^2$

$\frac{7}{32}\pi 4$

= 2.7489

$V_{actual} = \frac{1}{3}\pi \times 1 \times 2^2$

$\frac{1}{3}\pi 4$

= 4.189

Cone of height 5 and radius 15

$V_{approx} = \frac{7}{32}\pi \times 5 \times 15^2$

$\frac{7}{32}\pi \times 5 \times 225$

$= 773.13$

$V_{actual} = \frac{1}{3}\pi \times 5 \times 15^2$

$\frac{1}{3}\pi \times 5 \times 225$

$= 1178.1$

Cone of height 10 and radius of 20

$V_{approx} = \frac{7}{32}\pi \times 10 \times 20^2$

$\frac{7}{32}\pi \times 10 \times 400$

$= 2748.9$

$V_{actual} = \frac{1}{3}\pi \times 10 \times 20^2$

$\frac{1}{3}\pi \times 10 \times 400$

$= 4188.8$

Cylinders outside cones

Cone of height 1 and radius 2

$V_{approx} = \frac{15}{32}\pi \times 1 \times 2^2$

$\frac{15}{32}\pi \times 4$

$= 5.89$

$V_{actual} = 4.188$

Cone of height 5 and radius 15

$V_{approx} = \frac{15}{32}\pi \times 5 \times 15^2$

$\frac{15}{32}\pi \times 5 \times 225$

$= 1656.7$

$V_{actual} = 1178.1$

Cone of height 10 and radius 20

$V_{approx} = \frac{15}{32}\pi \times 10 \times 20^2$

$\frac{15}{32}\pi \times 10 \times 400$

$V_{actual} = 4188.8$

= 5890.5

These rules aren't very accurate for any size of cone.

By increasing the number of cylinders investigate how the formula for a cone may be discovered,

Formula for the volume of n number of cylinders inside a cone.

$$V_{3cylinders} = p\left(\frac{h}{4}\right)\left(\frac{r}{4}\right)^2 + p\left(\frac{h}{4}\right)\left(\frac{2r}{4}\right)^2 + p\left(\frac{h}{4}\right)\left(\frac{3r}{4}\right)^2$$

$$p\frac{h}{4}\left[\left(\frac{r}{4}\right)^2 + \left(\frac{2r}{4}\right)^2 + \left(\frac{3r}{4}\right)^2\right]$$

$$p\frac{h}{4}\left(\frac{r^2}{16} + \frac{4r^2}{16} + \frac{9r^2}{16}\right)$$

$$pr^2\frac{h}{4}\left(\frac{1}{16} + \frac{4}{16} + \frac{9}{16}\right)$$

$$\frac{1}{n+1}\left[\frac{\frac{1}{6}n(n+1)(2n+1)}{(n+1)^2}\right]phr^2$$

$$\frac{1}{n+1}\left[\frac{\frac{1}{6}n(2n+1)}{n+1}\right]phr^2$$

$$\frac{\frac{1}{6}n(2n+1)}{(n+1)^2}phr^2$$

Formula for the volume of n number of cylinders outside a cone.

$$V_{4cylinders} = p\left(\frac{h}{4}\right)\left(\frac{r}{4}\right)^2 + p\left(\frac{h}{4}\right)\left(\frac{2r}{4}\right)^2 + p\left(\frac{h}{4}\right)\left(\frac{3r}{4}\right)^2 + p\left(\frac{h}{4}\right)\left(\frac{4r}{4}\right)^2$$

$$p\frac{h}{4}\left[\left(\frac{r}{4}\right)^2 + \left(\frac{2r}{4}\right)^2 + \left(\frac{3r}{4}\right)^2 + \left(\frac{4r}{4}\right)^2\right]$$

$$\pi \frac{h}{4}\left(\frac{r^2}{16}+\frac{4r^2}{16}+\frac{9r^2}{16}+\frac{16r^2}{16}\right)$$

$$\frac{1}{4}\left(\frac{1}{16}+\frac{4}{16}+\frac{9}{16}+\frac{16}{16}\right)\pi hr^2$$

$$\frac{1}{n}\left[\frac{\frac{1}{6}n(n+1)(2n+1)}{n^2}\right]\pi hr^2$$

$$\frac{1}{n}\left[\frac{\frac{1}{6}(n+1)(2n+1)}{n}\right]\pi hr^2$$

$$\frac{\frac{1}{6}(n+1)(2n+1)}{n^2}\pi hr^2$$

As the value for n tends to infinity the value for $\frac{\frac{1}{6}(n+1)(2n+1)}{n^2}$ gets closer and closer to $\frac{1}{3}$ but will always be greater than $\frac{1}{3}$.

As the value for n tends to infinity the value for $\frac{\frac{1}{6}n(2n+1)}{(n+1)^2}$ gets closer and closer to $\frac{1}{3}$ but will always be less than $\frac{1}{3}$.

By making n a very high number like 1000000 and getting the values for $\frac{\frac{1}{6}(n+1)(2n+1)}{n^2}$ and $\frac{\frac{1}{6}n(2n+1)}{(n+1)^2}$ we see that these are both very close to $\frac{1}{3}$, by getting the average of these two numbers we can get an almost exact value and therefor hypothesize that the formula for the volume of a cone is indeed $\frac{1}{3}\pi r^2 h$.