RREF

1. Find the left most non-zero column.
2. Interchange the top row with another, if necessary, to bring a non-zero entry to the top of the column determined in step 1.
3. Add suitable multiples of the top row to the rows below so that all entries below the pivot are zero.
4. Now cover the top row and repeat from step 1 until matrix is in row-echelon form.
5. Starting with the last non-zero row, add suitable multiples of each row to the rows above to produce zero entries above the pivots. Multiply the rows by suitable constants to make the pivot values 1.

Linear Combinations

Given vectors $v_1, v_2, \cdots v_p$ in $\Re^n$, and scalars $c_1, c_2 \cdots c_p$, the vector $w = \sum_{i=1}^{p} c_i v_i$ is the **linear combination** of $v_1, v_2, \cdots v_p$ with **weights** $c_1, c_2 \cdots c_p$

$w = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ with weights 2 and 4

Span

The set of all linear combinations of $v_1, v_2, \cdots v_p$, is the subset of $\Re^n$ **spanned** by $v_1, v_2, \cdots v_p$, and is denoted $Span\ \{v_1, v_2, \cdots, v_p\}$

Vector Parametric Form

$$x_1 = -x_2 - x_5$$
$$x_2 = x_2$$
$$x_3 = -x_5$$
$$x_4 = 0$$
$$x_5 = x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

A system of equations is **homogeneous** if it is of the form $Ax = 0$.

Linear Independence

A set $\{v_1, \cdots, v_k\}$ of vectors in $\Re^m$ is **linearly independent** if $x_1 v_1 + \cdots + x_k v_k = 0$ is only true if all the weights $x_1, \cdots, x_k$ are zero.

The set $\{v_1, \cdots, v_k\}$ is **linearly dependent** if there are weights $x_1, \cdots, x_k$, *not all zero* such that $x_1 v_1 + \cdots + x_k v_k = 0$.

Linear Transformations

A transformation T: $\Re^n \to \Re^m$ is **linear** if: $T(u+v) = T(u) + T(v)$, (for all $u, v \in \Re^n$).
$T(cu) = cT(u)$, (for all $u \in \Re^n$

Subspace

A subspace of $\Re^n$ is any set $H \in \Re^n$ that has the following properties:

1. $0 \in H$.
2. For any $u, v \in H$ the sum $u + v \in H$.
3. For any $u \in H$, and $c \in \Re$, then $cu \in H$.

Inverse of a matrix

2x2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

if and only if $ad - bc \neq 0$.

Any, RR with identity matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right]$$

Determinants

Det(a)=0 if and only if Matrix A is singular (not invertible)

For 2x2 matrix

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

For 3x3 matrix

$$\begin{array}{ccccc} + & + & + & - & - & - \end{array}$$
$$\left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

The sum of the "right" pointing products subtract the "left" pointing products.
Can do determinants by cofactor expansion but it gets way too complicated!

Properties of determinants

Row operations

- If the matrix B is obtained from matrix A via a row replacement of the form $R_i \leftarrow R_i + cR_j$, then det(B)=det(A).
- If the matrix B is obtained from A by interchanging two rows, that is $R_i \leftrightarrow R_j$, then det(B)=(-1)det(A).
- If the matrix B is obtained from A by scaling one row of A. that is $R_i \leftarrow kR_i$, then det(B)=*k*det(A).

Determinant using row reduction

Reduce to echelon form (pivots not scaled to 1's). With *r* interchanges then $\det A = (-1)^r \det U$

$\begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \\ 0 & 0 & 0 & \bullet \end{bmatrix}$ If it is invertible the pivots are on the diagonal. detU is the product of the pivots.

Column Space

The column space of a matrix A is the set Col(A) of all linear combinations of the columns of A.
Col(A)= $span\{a_1, \cdots, a_n\}$
where $A = [a_1, \cdots a_n]$.
Col(A) is a subspace. **Eg.**

$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ then row reduce

$R = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ find the independent columns, the ones with the pivots.

A basis for Col(A) is $\left\{\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}\right\}$.

Null space

The null space of a matrix A is the set Null(A) of all solutions of the homogeneous equation $Ax = 0$.
Null(A) is a subspace. **Eg.**

$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ then row reduce

$R = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$x_1 = 2x_2 + x_4 - 3x_5$$
$$x_2 = x_2$$
$$x_3 = -2x_4 + 2x_5$$
$$x_4 = x_4$$
$$x_5 = x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Complex numbers

Basic facts

1. The set C contains an element called *i*, which is not a real number.
2. Every member of C can be written uniquely in the form: $x + iy$. Where *x* and *y* are real numbers.
3. The operations of addition, negation and multiplication are defined:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$
$$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$
$$-(x_1 + iy_1) = (-x_1 + i(-y_1))$$
$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$
$$i = \sqrt{-1}$$

inverse of non-zero complex number:

$$(x + iy)^{-1} = \left(\frac{x}{x^2 + y^2}\right) - i\left(\frac{y}{x^2 + y^2}\right)$$
$$(x + iy)(x - iy) = x^2 + y^2$$

Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Finding roots of equations

Newton's

Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Fixed point

iteration: $x_{n+1} = f(x_n)$

Curve Sketching

1. Calculate $f'(x)$ and $f''(x)$, and express the results in factored form.
2. Examine $f(x)$ to determine domain and following items.
   (a) Any vertical asymptotes. (look for zeroes of denominators)
   (b) Any horizontal asymptotes. (consider $\lim_{x\to\pm\infty} f(x)$.)
   (c) Any obvious symmetry. (is f even or odd?)
   (d) Any easily calculated intercepts (points with coords $(x,0)$ or $(0,y)$, or endpoints or "obvious" points.
3. Examine $f'(x)$ for the following:
   (a) Any critical points.
   (b) Any points where $f'(x)$ is not defined. (Including singular points, endpoints of the domain of f, vertical asymptotes.
   (c) Intervals on which f' is positive or negative. Put in table. Conclusions about where f is increasing and decreasing and classification of some critical and singular points, local maxima, minima, all on chart, table.
4. Examine $f''(x)$ for the following:
   (a) Points where $f''(x) = 0$, inflexion.
   (b) Points where $f''(x)$ is undefined. (singular points, endpoints, vertical asymptotes)
   (c) Intervals where $f''(x)$ is positive or negative and f is therefore concave up or down. + = concave up, - = concave down.
   (d) Any inflexion points.

Linear Approximation

The linear approximation of the function f about $x = a$ is the function: $L(x) = f(a) + f'(a)(x-a)$

Error: $E(x) = f(x) - L(x) = \dfrac{f''(x)}{2}\cdot(x-a)^2$

Corollary A:

If $f''(x) > 0$ between a and x then $f(x) > L(x)$

If $f''(x) < 0$ between a and x then $f(x) < L(x)$

Corollary B:

If $|f''(t)| < K$ for all t between a and x,

then $|E(x)| < \dfrac{K}{2}(x-a)^2$

Corollary C:

If $M < f''(t) < N$ for all t between a and x then:

$$\frac{M}{2}(x-a)^2 < \frac{f''(t)}{2}(x-a)^2 < \frac{N}{2}(x-a)^2$$

$$\frac{M}{2}(x-a)^2 < E(x) < \frac{N}{2}(x-a)^2$$

$$L(x)+\frac{M}{2}(x-a)^2 < L(x)+E(x) < L(x)+\frac{N}{2}(x-a)^2$$

$$L(x)+\frac{M}{2}(x-a)^2 < f(x) < L(x)+\frac{N}{2}(x-a)^2$$

Better estimate: $f(x) \approx L(x) + \dfrac{M+N}{4}(x-a)^2$

Taylor Polynomials

Approximations. Degree 1 is good, degree 2 is better, degree n is fucking awesome.

Polynomial degree 1: $P_1(x) = L(x) = f(a) + f'(a)(x-a)$

Polynomial degree 2: $P_2(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2}(x-a)^2$

In general if $f^{(n)}(x)$ exists in an open interval containing $x = a$, then the polynomial

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

matches f and it's first n derivatives at $x = a$. $P_n^{(n)}(a) = f^{(n)}(a)$

This describes $f(x)$ near $x = a$ better than any other polynomial of degree n.

Taylor's Formula

If the $(n+1)$st-order derivative, $f^{(n+1)}(t)$, exists for all t in an interval containing a and x and if $P_n(x)$ is the Taylor polynomial of degree n for $f(x)$ about $x = a$ then the formula $f(x) = P_n(x) + E_n(x)$ holds where the error term $E_n(x)$ (LaGrange Remainder) is given by:

$E_n(x) = \dfrac{f^{(n+1)}(X)}{(n+1)!}(x-a)^{n+1}$ where X is some number between a and x.

Big-O Notation

$f(x) = O(u(x))$ as $x \to a$ provided that $|f(x)| \le K|u(x)|$ holds for some constant K on some open interval containing $x = a$.

Similarly, $f(x) = g(x) + O(u(x))$ as $x \to a$ if $f(x) - g(x) = O(u(x))$ as $x \to a$, that is, if: $|f(x) - g(x)| \le K|u(x)|$ near a.

Differentiation Rules

$$\frac{d}{dx}(f(x)+g(x))=f'(x)+g'(x)$$

$$\frac{d}{dx}(cf(x))=cf'(x)$$

$$\frac{d}{dx}(f(x)g(x))=f'(x)g(x)+f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right)=-\frac{f'(x)}{(f(x))^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

$$\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Elementary Derivatives

$$\frac{d}{dx}\frac{1}{x}=-\frac{1}{x^2}$$

$$\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}x^r=rx^{r-1}$$

$$\frac{d}{dx}e^x=e^x$$

$$\frac{d}{dx}a^x=a^x\ln a$$

$$\frac{d}{dx}\ln x=\frac{1}{x}$$

$$\frac{d}{dx}\sin x=\cos x$$

$$\frac{d}{dx}\cos x=-\sin x$$

$$\frac{d}{dx}\tan x=\sec^2 x$$

$$\frac{d}{dx}\sec x=\sec x\cdot\tan x$$

$$\frac{d}{dx}\csc x=-\csc x\cdot\cot x$$

$$\frac{d}{dx}\cot x=-\csc^2 x$$

$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$$

$$\frac{d}{dx}|x|=\operatorname{sgn}x=\frac{x}{|x|}$$

Laws of Exponents

If $a>0$ and $b>0$, and x and y are any real numbers then,

$$a^0=1$$

$$a^{x+y}=a^x\cdot a^y$$

$$a^{-x}=\frac{1}{a^x}$$

$$a^{x-y}=\frac{a^x}{a^y}$$

$$\left(a^x\right)^y=a^{xy}$$

$$(ab)^x=a^x\cdot b^x$$

Trigonometric Identities

$$\sin^2 x+\cos^2 x=1$$

$$\sin(-x)=-\sin x$$

$$\cos(-x)=\cos x$$

$$\sec^2 x=1+\tan^2 x$$

$$\sin(\pi-x)=\sin x$$

$$\cos(\pi-x)=-\cos x$$

$$\csc^2 x=1+\cot^2 x$$

$$\sin\left(\frac{\pi}{2}-x\right)=\cos x$$

$$\cos\left(\frac{\pi}{2}-x\right)=\sin x$$

$$\sin(x\pm y)=\sin x\cdot\cos y\pm\cos x\cdot\sin y$$

$$\cos(x\pm y)=\cos x\cdot\cos y\mp\sin x\cdot\sin y$$

$$\tan(x\pm y)=\frac{\tan x\pm\tan y}{1\mp\tan x\tan y}$$

$$\sin 2x=2\sin x\cdot\cos x$$

$$\cos 2x=2\cos^2 x-1=1-2\sin^2 x$$

$$\sin^2 x=\frac{1-\cos 2x}{2}$$

$$\cos^2 x=\frac{1+\cos 2x}{2}$$

Quadratic Formula

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

Laws of Logarithms

If

$a>0$ and $b>0$ and $a\neq 1$ and $b\neq 1$ then,

$$\log_a 1=0 \qquad a^0=1$$

$$\log_a(xy)=\log_a x+\log_a y$$

$$\log_a\left(\frac{1}{x}\right)=-\log_a x$$

$$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$$

$$\log_a\left(x^y\right)=y\log_a x$$

$$\log_a x=\frac{\log_b x}{\log_b a}$$

Limit Rules

If $\lim_{x\to a}f(x)=L$ and $\lim_{x\to a}g(x)=M$ and k is a constant, then

Limit of a sum: $\lim_{x\to a}[f(x)+g(x)]=L+M$

Limit of a difference: $\lim_{x\to a}[f(x)-g(x)]=L-M$

Limit of a product: $\lim_{x\to a}f(x)g(x)=L\cdot M$

Limit of a multiple: $\lim_{x\to a}k\cdot f(x)=k\cdot L$

Limit of a quotient: $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{L}{M}$ if $M\neq 0$

Limit of a power: $\lim_{x\to a}(f(x))^{\frac{m}{n}}=L^{\frac{m}{n}}$

Order is preserved $L\le M$ if $f(x)\le g(x)$

More Big-O stuff

<u>Properties of the O notation</u>

1. If $f(x)=O(u(x))$ as $x\to a$, then $Cf(x)=O(u(x))$ as $x\to a$ for any constant c.
2. If $f(x)=O(u(x))$ as $x\to a$ and $g(x)=O(u(x))$ as $x\to a$ then: $f(x)\pm g(x)=O(u(x))$ as $x\to a$.
3. If $f(x)=O\left((x-a)^k u(x)\right)$ as $x\to a$ then: $\frac{f(x)}{(x-a)^k}=O(u(x))$ as $x\to a$ for any constant k.

Taylor's Theorem says that if $f^{(n+1)}(t)$ exists on an interval containing x and a, and if $P_n(x)$ is the Taylor Polynomial for $f(x)$ about $x=a$, then, as $x\to a$:

$$f(x)=P_n(x)+O\left((x-a)^{n+1}\right).$$

If $f(x)=Q_n(x)+O\left((x-a)^{n+1}\right)$ as $x\to a$ where $Q_n(x)$ is a polynomial of degree at most n, then $Q_n(x)=P_n(x)$, the Taylor polynomial of $f(x)$ about $x=a$